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DIFFRACTION OF AN ANTIPLANE SHEAR WAVE BY TWO COPLANAR GRIFFITH CRACKS IN AN INFINITE ELASTIC MEDIUM

S. Itou
Department of Mechanical Engineering, Hachinohe Institute of Technology, Hachinohe 031, Japan

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Abstract—The scattering of a time-harmonic antiplane shear wave by two parallel and coplanar Griffith cracks embedded in an infinite elastic medium is considered. The input wave normally impinges on the cracks. Fourier transformations are utilized to reduce the problem to two simultaneous integral equations which can be solved by the series expansion method. The dynamic stress intensity factors are numerically computed.

1. INTRODUCTION

From an engineering viewpoint, there are two reasons why dynamic crack problems are of interest. One of them is that the dynamic stress intensity factors may be about 30% higher than the corresponding static ones [1, 2]. Another reason is that the dynamic fracture toughness value is considerably lower than the static fracture toughness value [3]. As regards the dynamic crack problem, research has been restricted to the case of a single crack because of the severe mathematical complexity encountered in finding solutions for two or more cracks. However, recently, Jain and Kanwal have overcome this difficulty and have presented their solution for the diffraction problem for normally incident longitudinal and antiplane shear waves by two symmetrical coplanar Griffith cracks located in an infinite isotropic and homogeneous elastic medium [4]. They reduced the problem to the solution of an integro-differential equation and expanded it in a power series of the wave number which is assumed to be small. The integral equations were solved in a manner similar to that employed by Lowengrub and Srivastav to solve the corresponding static problem [5]. Jain and Kanwal's solutions are approximate ones which are valid at the low-frequency end, hence, the maximum values of the stress intensity factors were not verified. A similar problem related to the P wave has been worked over using a different approach [6].

Recently, the author noticed that there might be a mistake in the numerical results of the stress intensity factors for an incident SH wave given by Jain and Kanwal [4], because the following tendency cannot be observed; namely, at the low-frequency end of the frequency scale, the stress intensity factor generally increases as the wave number increases [1, 2].

In this paper, the stresses in an infinite elastic plate weakened by two coplanar Griffith cracks are determined for a normally incident horizontally polarized shear wave. The analysis is time-harmonic, so the wavefront effects do not appear. The boundary condition equation for the problem is reduced to dual integral equations by means of the Fourier transformations. In an effort to solve the equations, the author, utilizing the work carried out in his previous paper [6], expands the surface displacement in a series of a function which is automatically zero outside of the cracks. By doing this, the integral equations can be easily solved by the Schmidt method. The key to the solution is quite simple, however, the quality of the solution is equivalent to those given by Sih *et al.* [1, 2]. Numerical calculations are carried out for the dynamic stress intensity factors.

2. FUNDAMENTAL EQUATIONS

We consider the cracks to be placed along the x_1-axis from $-b$ to $-a$ and a to b with reference to a set of a rectangular coordinates x_1, x_2 and x_3 as shown in Fig. 1. The x_3-axis is directed parallel to the crack's edges such that the only nonvanishing displacement is the x_3 direction component, $u_3^* = u_3^*(x_1, x_2, t)$. In the absence of body forces, the equation of motion is

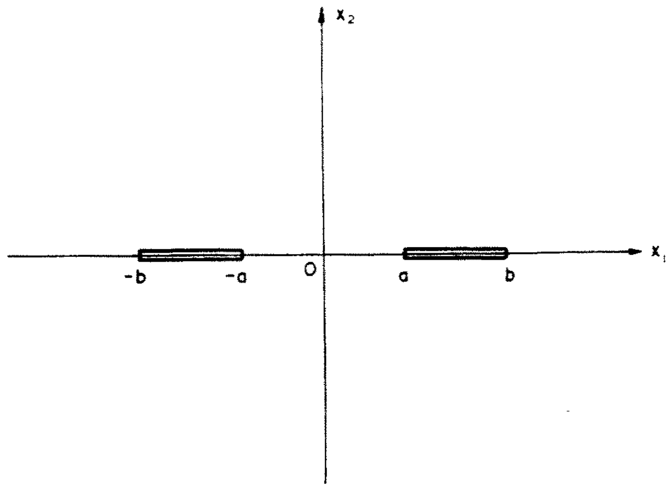

Fig. 1. Geometry and coordinate system.

given assuming the medium to be homogeneous and isotropic,

$$u^*_{3,11} + u^*_{3,22} = 1/c_2{}^2 \cdot \partial^2 u^*_3/\partial t^2, \tag{1}$$

where $c_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity, μ is the shear modulus, ρ is the density of the material, t is time and the indices following a comma indicate the partial differentiation with respect to the variable, e.g. $u^*_{3,1} = \partial u^*_3/\partial x_1$. The nonvanishing stress components are expressed as

$$\begin{aligned} \tau^*_{13} &= \mu u^*_{3,1}, \\ \tau^*_{23} &= \mu u^*_{3,2}. \end{aligned} \tag{2}$$

Assuming the solution of eqn (1) exhibits harmonic time-dependence, we may write

$$u^*_3 = u_3 \exp(-i\omega t), \tag{3}$$

where ω is the circular frequency. Substituting eqn (3) into (1) we obtain

$$u_{3,11} + u_{3,22} + \alpha^2 u_3 = 0, \tag{4}$$

with

$$\alpha = \omega/c_2. \tag{5}$$

The incident antiplane displacement wave which impinges on the line cracks at right angle to the x_1-axis can be expressed as

$$u^{*(i)}_3 = \epsilon_0 \exp\{-i(\alpha x_2 + \omega t)\}, \tag{6}$$

where ϵ_0 is the amplitude. Thus, we obtain the nonvanishing component of the incident stress fields, dropping the time factor $\exp(-i\omega t)$

$$\tau^{(i)}_{23} = i\tau_0 \exp(-i\alpha x_2), \tag{7}$$

with

$$\tau_0 = -\alpha\mu\epsilon_0. \tag{8}$$

The boundary conditions for the scattered field are

$$u_3{}^{(s)}(x_1, 0) = 0, \quad \text{for} \quad |x_1| > b, \quad |x_1| < a, \tag{9a}$$

$$\tau^{(s)}_{23}(x_1, 0) = -i\tau_0, \quad \text{for} \quad a < |x_1| < b. \tag{9b}$$

3. ANALYSIS

The antiplane character of the displacement field requires $u_3^{(s)}$ to be odd in x_2 and the scattered field may be represented as

$$u_3^{(s)} = \operatorname{sgn}(x_2)\frac{2}{\pi}\int_0^\infty A(\xi)\exp(-\gamma|x_2|)\cos(\xi x_1)\,\mathrm{d}\xi, \tag{10}$$

where

$$\begin{aligned} \operatorname{sgn}(x_2) &= -1, \quad \text{for} \quad x_2 < 0, \\ \operatorname{sgn}(x_2) &= 0, \quad \text{for} \quad x_2 = 0, \\ \operatorname{sgn}(x_2) &= 1, \quad \text{for} \quad x_2 > 0, \end{aligned} \tag{11}$$

and

$$\gamma = (\xi^2 - \alpha^2)^{1/2} = -i(\alpha^2 - \xi^2)^{1/2}. \tag{12}$$

Because of the symmetry conditions in eqn (9), it is possible to consider the problem for the half-plane, $x_2 \geqq 0$, only. Substituting eqn (10) into the stress expressions, we obtain

$$\begin{aligned} \tau_{13}^{(s)} &= -\frac{2\mu}{\pi}\int_0^\infty \xi A(\xi)\exp(-\gamma x_2)\sin(\xi x_1)\,\mathrm{d}\xi, \\ \tau_{23}^{(s)} &= -\frac{2\mu}{\pi}\int_0^\infty \gamma A(\xi)\exp(-\gamma x_2)\cos(\xi x_1)\,\mathrm{d}\xi. \end{aligned} \tag{13}$$

We take the following series as the surface displacement $u_3^{(s)}(x_1, 0)$,

$$\begin{aligned} u_3^{(s)}(x_1, 0) = \frac{\tau_0}{\mu}\sum_{n=1}^{\infty} q_n \Bigg[\!\!\Bigg[& \frac{1}{2n}\cos(n\pi/2)\sin[n\sin^{-1}\{(a+b-2|x_1|)/(b-a)\}] \\ & -\frac{1}{2n}\sin(n\pi/2)\cos[n\sin^{-1}\{(a+b-2|x_1|)/(b-a)\}]\Bigg]\!\!\Bigg], \\ & \text{for} \quad a < |x_1| < b, \\ = 0, \qquad & \text{for} \quad 0 \leqq |x_1| < a, \\ & \qquad\;\; b < |x_1| < \infty. \end{aligned} \tag{14}$$

Integral expression of eqn (14) is,

$$u_3^{(s)}(x_1, 0) = \frac{\tau_0}{\mu}\sum_{n=1}^{\infty} q_n \int_0^\infty \frac{1}{\xi}\sin\{(a+b)\xi/2 - n\pi/2\}J_n\{(b-a)\xi/2\}\cos(\xi x_1)\,\mathrm{d}\xi, \tag{15}$$

where q_n are the unknown coefficients to be determined and $J_n(\,)$ are Bessel functions. Then, the unknown function $A(\xi)$ is obtained so as to satisfy the boundary condition (9a)

$$A(\xi) = (\tau_0\pi)/(2\mu)\sum_{n=1}^{\infty} q_n \frac{1}{\xi}\sin\{(a+b)\xi/2 - n\pi/2\}J_n\{(b-a)\xi/2\}. \tag{16}$$

The remaining boundary condition (9b) can be reduced to

$$\sum_{n=1}^{\infty} q_n \left[\int_0^\infty (-\gamma)/\xi \sin\{(a+b)\xi/2 - n\pi/2\}J_n\{(b-a)\xi/2\}\cos(\xi x_1)\,\mathrm{d}\xi\right] = -i, \quad \text{for} \quad a < |x_1| < b. \tag{17}$$

To evaluate numerically the semi-infinite integrals in eqn (17), we must replace the integrand so as to achieve rapid decay when ξ becomes large. For this purpose, the following relationship is used,

$$\int_0^\infty \cos(a_1\xi)J_n(a_2\xi)\,d\xi = \cos(n\epsilon)/\sqrt{(a_2^2-a_1^2)}, \quad \text{for} \quad a_2>a_1>0,$$

$$= -a_2^n \sin(n\pi/2)/\{\sqrt{(a_1^2-a_2^2)}(a_1+\sqrt{(a_1^2-a_2^2)})^n\},$$

$$\int_0^\infty \sin(a_1\xi)J_n(a_2\xi)\,d\xi = \sin(n\epsilon)/\sqrt{(a_2^2-a_1^2)}, \quad \text{for} \quad a_2>a_1>0,$$

$$= a_2^n \cos(n\pi/2)/\{\sqrt{(a_1^2-a_2^2)}(a_1+\sqrt{(a_1^2-a_2^2)})^n\}, \quad \text{for} \quad a_1>a_2>0, \tag{18}$$

with

$$\epsilon = \sin^{-1}(a_1/a_2). \tag{19}$$

Therefore, the integral can be rewritten as

$$\int_0^\infty (-\gamma)/\xi \sin\{(a+b)\xi/2-n\pi/2\}J_n\{(b-a)\xi/2\}\cos(\xi x_1)\,d\xi$$

$$= \int_0^\infty (\xi-\gamma)/\xi \Big[\!\Big[\frac{1}{2}\cos(n\pi/2)[\sin\{(a+b+2x_1)\xi/2\}+\sin\{(a+b-2x_1)\xi/2\}]$$

$$-\frac{1}{2}\sin(n\pi/2)[\cos\{(a+b-2x_1)\xi/2\}+\cos\{(a+b+2x_1)\xi/2\}]\Big]\!\Big]$$

$$\times J_n\{(b-a)\xi/2\}\,d\xi - \frac{1}{2}\{(b-a)/2\}^n/[\![[\{(a+b+2x_1)/2\}^2-\{(b-a)/2\}^2]^{1/2}$$

$$\times[\![(a+b+2x_1)/2+[\{(a+b+2x_1)/2\}^2-\{(b-a)/2\}^2]^{1/2}]\!]^n]\!]+\frac{1}{2}\sin[\![n[\pi/2-\sin^{-1}$$

$$\times\{(a+b-2x_1)/(b-a)\}]]\!]/[\{(b-a)/2\}^2-\{(a+b-2x_1)/2\}^2]^{1/2}, \quad \text{for} \quad a<x_1<b. \tag{20}$$

The function $(\xi-\gamma)/\xi$ behaves as ξ^{-2} for large ξ, so that the semi-infinite integral on the r.h.s. of eqn (20) can easily be evaluated numerically by Filon's method [7].

Equation (17) can be solved for coefficients q_n by the Schmidt method [8].

4. STRESS INTENSITY FACTORS

As coefficients q_n are known the entire stress field can be given. However, for the crack problem, stress τ_{23} just ahead of the crack is important. Stress $\tau_{23}^{(s)}$ at $x_2=0$ is given as

$$\tau_{23}^{(s)}(x_1,0)/\tau_0 = \sum_{n=1}^\infty q_n \int_0^\infty (\xi-\gamma)/\xi \sin\{(a+b)\xi/2-n\pi/2\}J_n\{(b-a)\xi/2\}\cos(\xi x_1)\,d\xi$$

$$-\sum_{n=1}^\infty q_n \int_0^\infty J_n\{(b-a)\xi/2\}\Big[\!\Big[\frac{1}{2}\cos(n\pi/2)[\sin\{(a+b+2x_1)\xi/2\}+\sin\{(a+b-2x_1)\xi/2\}]$$

$$-\frac{1}{2}\sin(n\pi/2)[\cos\{(a+b-2x_1)\xi/2\}+\cos\{(a+b+2x_1)\xi/2\}]\Big]\!\Big]\,d\xi. \tag{21}$$

Thus, the stress intensity factors at the inner and outer edges of the crack are defined by the expressions

$$K_i = \lim_{x_1\to a-} \tau_{23}^{(s)}(x,0)\sqrt{(2\pi(a-x_1))} = -\tau_0\sqrt{(\pi/\{2(b-a)\})}\sum_{n=1}^\infty q_n,$$

$$K_0 = \lim_{x_1\to b+} \tau_{23}^{(s)}(x_1,0)\sqrt{(2\pi(x_1-b))} = \tau_0\sqrt{(\pi/\{2(b-a)\})}\sum_{n=1}^\infty (-1)^n q_n. \tag{22}$$

5. NUMERICAL EXAMPLES AND RESULTS

Adopting the first seven terms of the infinite series in eqn (17), the author used Schmidt's procedures. For an accuracy check, the values of $\sum_{n=1}^{7} q_n E_n(x_1)$ and $-u(x_1)$ are given in Table 1 for $\alpha(b-a)/2 = 0.5$, 1.5 and $a/b = 0.5$. From this it is clear that the Schmidt method is carried out satisfactorily. In Table 2, the values of q_n are shown for $\alpha(b-a)/2 = 0.5$, 1.5 and $a/b = 0.5$. Table 3 shows the ratios of the peak stress intensity factors $|K_i|$ and $|K_0|$ to the corresponding static ones, K_i^s and K_0^s. In Figs. 2 and 3, the absolute values of K_0 and K_i are plotted for

Table 1. The values of $\sum_{n=1}^{7} q_n E_n(x_1)$ and $-u(x_1)$ for $\alpha(b-a)/2 = 0.5$, 1.5 and $a/b = 0.5$

$\alpha(b-a)/2$	x_1/b	$\sum_{n=1}^{7} q_n E_n(x_1)$	$-u(x_1)$
0.5	0.50000	-0.89454×10^{-6} $-0.99999i$	-1.0i
	0.53571	0.82738×10^{-6} $-1.00000i$	
	⋮	⋮	
	0.75000	-0.17540×10^{-6} $-1.00000i$	
	⋮	⋮	
	0.96429	0.14835×10^{-6} $-1.00000i$	
	1.00000	-0.13913×10^{-5} $-0.99999i$	
1.5	0.50000	-0.40411×10^{-4} $-0.99997i$	-1.0i
	0.53571	0.41440×10^{-4} $-1.00000i$	
	⋮	⋮	
	0.75000	-0.57526×10^{-5} $-1.00000i$	
	⋮	⋮	
	0.96429	0.75486×10^{-4} $-1.00000i$	
	1.00000	-0.72543×10^{-4} $-0.99996i$	

Table 2. The values of q_n for $\alpha(b-a)/2 = 0.5$, 1.5 and $a/b = 0.5$

$\alpha(b-a)/2$	n	a/b = 0.5
0.5	1	0.77364×10^{-1} -0.55954×10^{-0} i
	2	0.49086×10^{-2} -0.81148×10^{-3} i
	3	-0.15321×10^{-2} $+0.90939\times10^{-2}$ i
	4	-0.39237×10^{-4} -0.40340×10^{-4} i
	5	0.78403×10^{-6} -0.48029×10^{-4} i
	6	0.23990×10^{-6} -0.31176×10^{-6} i
	7	0.13181×10^{-5} -0.95185×10^{-5} i
1.5	1	0.50625×10^{-0} -0.30469×10^{-0} i
	2	0.15487×10^{-1} -0.98477×10^{-2} i
	3	-0.90627×10^{-1} $+0.17448\times10^{-1}$ i
	4	-0.20990×10^{-2} $+0.73394\times10^{-3}$ i
	5	0.30889×10^{-2} -0.21945×10^{-3} i
	6	0.64345×10^{-4} -0.87807×10^{-6} i
	7	0.16147×10^{-4} -0.38845×10^{-4} i

Table 3. The absolute peak values of the dynamic stress intensity factors K_i and K_0

a/b	0.1	0.2	0.5	0.8	0.9
$\lvert K_i\rvert^{peak}/K_i^s$	1.21	1.17	1.29	1.28	1.28
$\lvert K_0\rvert^{peak}/K_0^s$	1.14	1.06	1.33	1.29	1.28

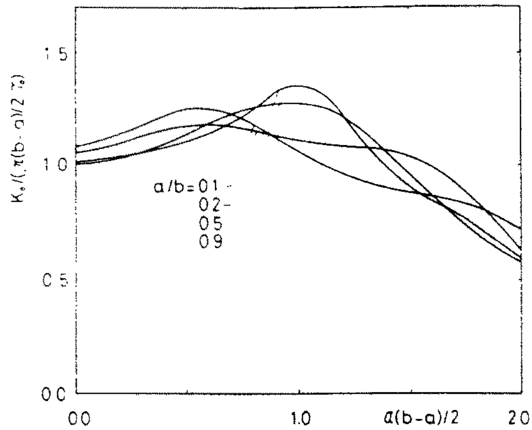

Fig. 2. Absolute value of the stress intensity factor at the outer edge of the crack vs $\alpha(b-a)/2$.

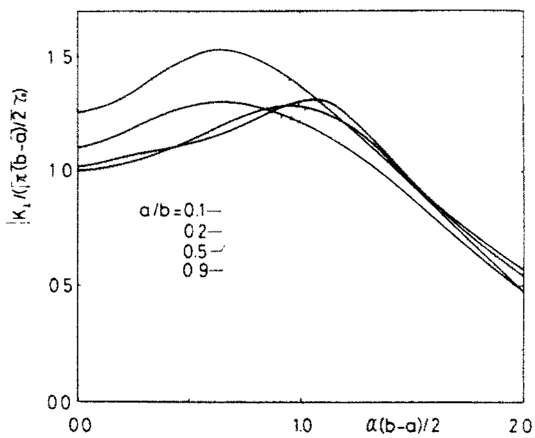

Fig. 3. Absolute value of the stress intensity factor at the inner edge of the crack vs $\alpha(b-a)/2$.

$a/b = 0.1$, 0.2, 0.5, 0.9 against the normalized wave number $\alpha(b-a)/2$. The results of both K_0 and K_i for $a/b = 0.9$ agree well with that given by Loeber and Sih for a single crack[1]. It is considered that Jain and Kanwal made a careless mistake in their numerical calculations of the stress intensity factors for an incident SH wave[4].

The difference between the present approach to the problem and that used in Ref. [4] is in the solving of the dual integral equations. Jain and Kanwal's solutions are effective at the low-frequency end of the frequency scale, and thereby, the series expansion method employed here is excellent due to its validity at low and intermediate frequencies.

From the numerical calculations, the author draws the following conclusions:

(i) For the static case, the stress intensity factor at the inner edge of the cracks is always larger than that of the outer one. However, the peak value of the dynamic stress intensity factor $|K_0|$ exceeds the other one $|K_i|$ for a given value of a/b such as 0.5.

(ii) At low frequencies such as $\alpha(b-a)/2 = 0.0 \sim 1.0$, $|K_i|$ increases rapidly as the value of a/b increases slightly for its small value.

(iii) It is considered that the maximum values of the stress intensity factors $|K_i|$ and $|K_0|$ are at most 35% greater than the corresponding static factors.

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